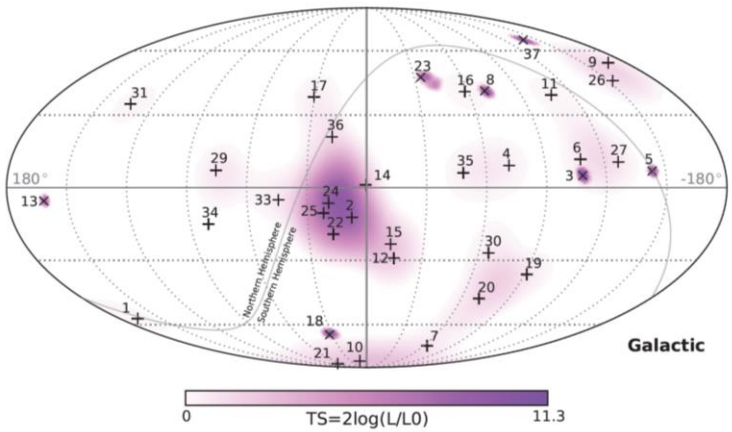

On Observing Anisotropy of Cosmic Neutrinos

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IceCube Particle Astrophysics Symposium
May 5, 2015

CCAPP
The Ohio State University's Center for Cosmology and AstroParticle Physics

THE OHIO STATE UNIVERSITY

Observing "Points" in the Sky

- High-Energy Radiation Events
  - Gamma-Rays
  - Cosmic Ray Shower Events
  - Cosmic Neutrinos

Inference radiation sources, cosmic ray acceleration, ray propagation, etc.

- Celestial Objects
  - Galaxies
  - AGN
  - X-ray Clusters
  - …

Inference cosmic expansion history, large scale structure, galaxy formation, etc.

Potential radiation sources!

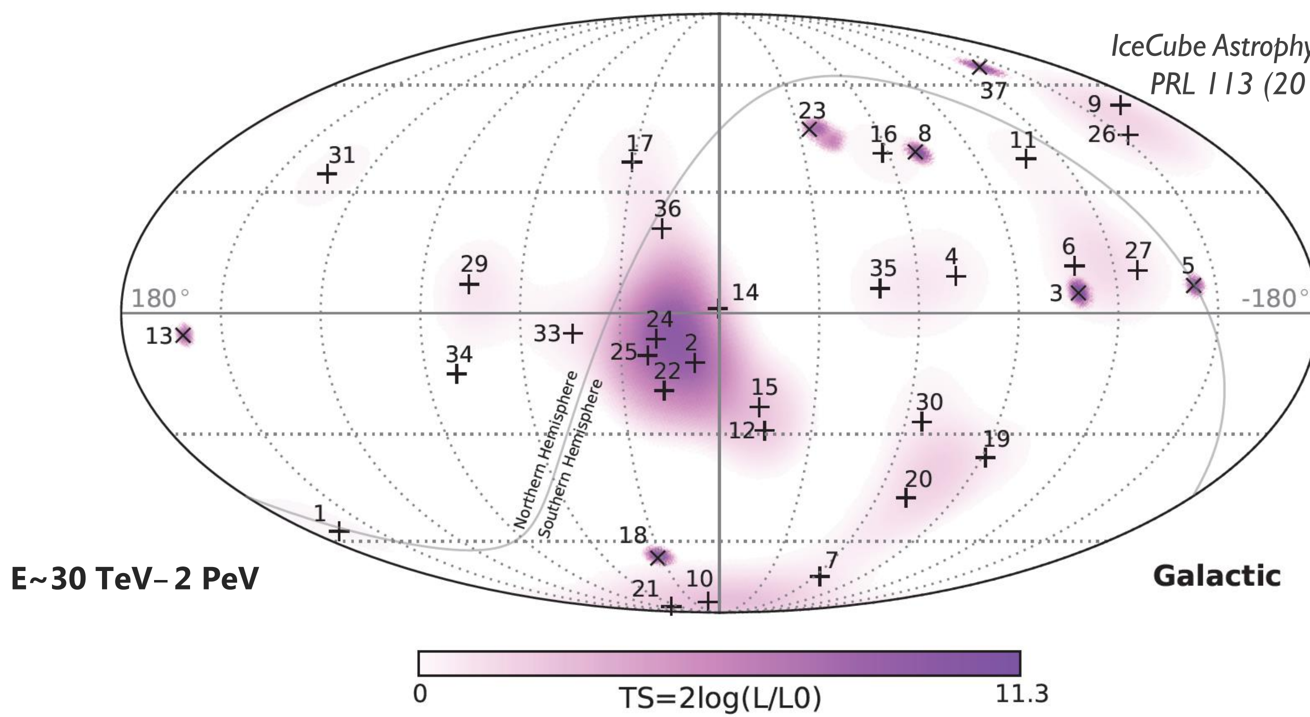
Specify distribution of a class of events/objects in the sky.

- objects in a redshift range, radiation events in an energy bin, etc.

Angular Distribution Methods

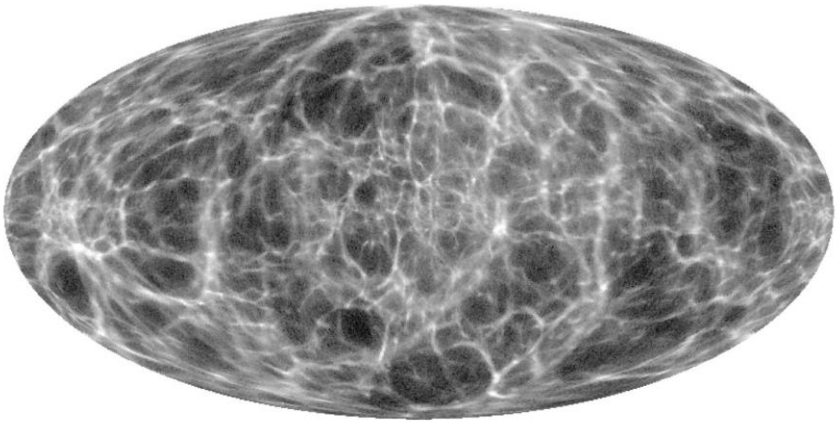
- When point sources cannot be resolved,
  - the angular distribution of **observed events** approaches the angular distribution of **sources** (messenger-propagated and projected) on our sky (full **skymap**).

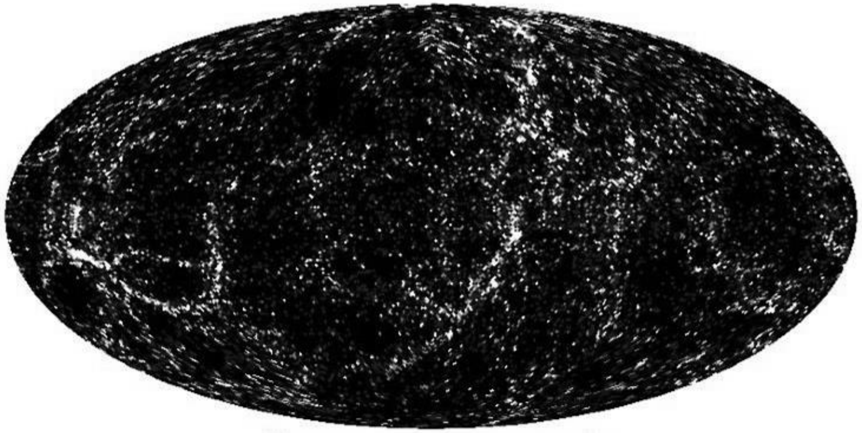
IceCube Astrophysical Neutrinos
PRL 113 (2014) 101101



Distinguishing Dense vs. Sparse

Francisco-Shu Kitaura et al., MNRAS 427, L35 (2012)





Dense Distributions, e.g.,

- radio galaxies
- dark matter annihilation

All events from different source.

Sparse Distributions, e.g.,

- active galactic nuclei
- local extragalactic structure

More sources with multiple events.

Given N events, what can we infer about the full **skymap**?

Given physical source models, how many events would distinguish them?

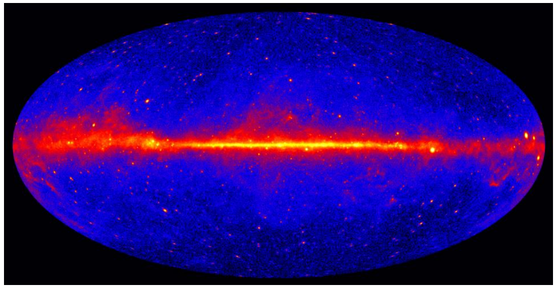
Angular Clustering of the Source Skymap

- Positive, real function on the sphere $F(\boldsymbol{n})$.
- Normalize: Let $S(\boldsymbol{n}) = \frac{F(\boldsymbol{n})}{\langle F \rangle} - 1.$

For cosmic neutrinos, F is the apparent flux map of all sources.

- Normalized spherical transform:

$$\tilde{a}_{\ell m} = \int d\boldsymbol{n}\, Y^*_{\ell m}(\boldsymbol{n}) S(\boldsymbol{n})$$



- Angular power spectrum:

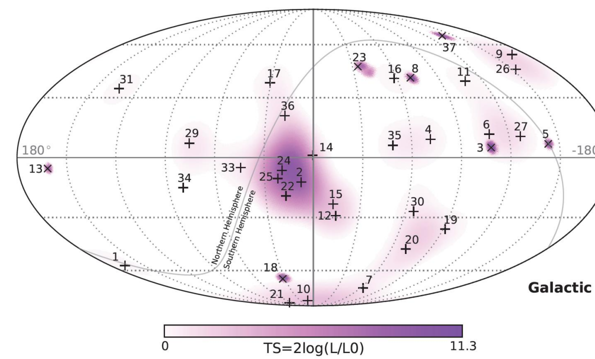
$$\tilde{C}_\ell = \frac{1}{2\ell+1} \sum_{m=-\ell}^{\ell} |\tilde{a}_{\ell m}|^2 = 4\pi \int \frac{d\boldsymbol{n}_1}{4\pi} \frac{d\boldsymbol{n}_2}{4\pi}\, S(\boldsymbol{n}_1) P_\ell(\boldsymbol{n}_1 \cdot \boldsymbol{n}_2)\, S(\boldsymbol{n}_2)$$

- Angular bispectrum:

$$\tilde{B}_{\ell_1 \ell_2 \ell_3} = \sum_{m_1, m_2, m_3} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \tilde{a}_{\ell_1 m_1} \tilde{a}_{\ell_2 m_2} \tilde{a}_{\ell_3 m_3}$$

Angular Clustering of Observed Events

- Differential flux of events $F_N(\boldsymbol{n}) = \frac{4\pi}{\varepsilon}\sum_{i=1}^{N}\delta(\boldsymbol{n}-\boldsymbol{n}_i)$.
  - Each term needs weights if exposure ε is not uniform.



- Normalize: $S_N(\boldsymbol{n}) = \frac{4\pi}{N}\sum_{i=1}^{N}\delta(\boldsymbol{n}-\boldsymbol{n}_i) - 1$.

- Normalized spherical transform:

$$\tilde{a}_{\ell m,N} = \frac{4\pi}{N}\sum_{i=1}^{N} Y_{\ell m}^{*}(\boldsymbol{n}_i)$$

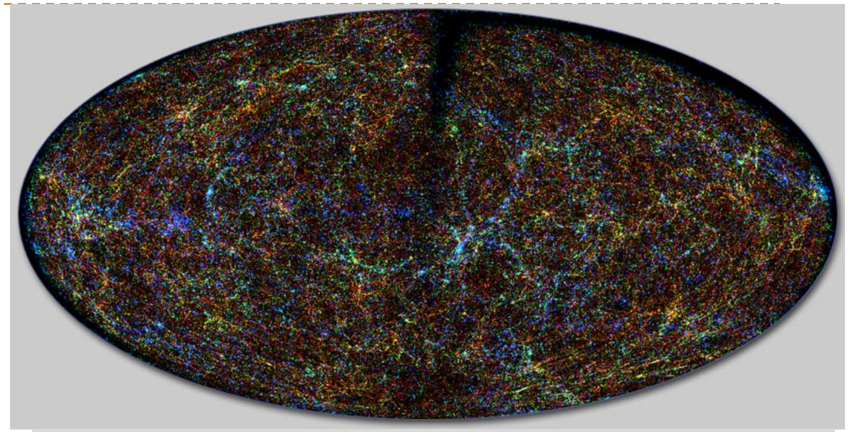
- Angular power spectrum of N events:

$$\tilde{C}_{\ell,N} = \frac{4\pi}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} P_\ell(\boldsymbol{n}_i \cdot \boldsymbol{n}_j)$$

Statistical properties of this observable tell us about the sources.

The Problem

- Let $\tilde{C}_\ell$ be the fluctuation (normalized) APS of a **skymap**– what we are trying to measure.
- Receive N events at random, weighted by the sky map.
- Assume full sky observations with uniform exposure.



A hypothetical projected **skymap** of sources.
The 2 micron sky courtesy of the 2MASS collaboration, http://www.ipac.caltech.edu/2mass/.

- What is the angular power spectrum of the N events, $\tilde{C}_{\ell,N}$, from a full sky map with distribution $\tilde{C}_\ell$?
  - mean of $\tilde{C}_{\ell,N}$?
  - variance of $\tilde{C}_{\ell,N}$?

Simplest Model: Poisson Point Process

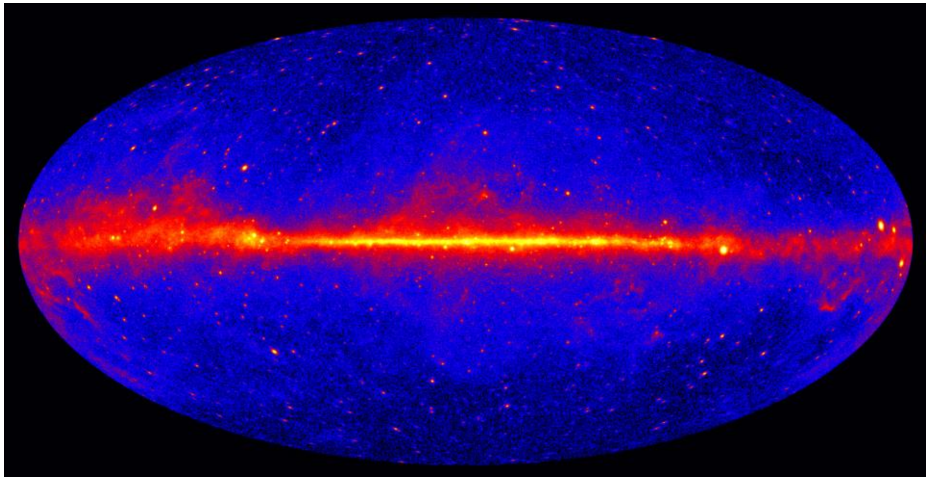
- Only 2 assumptions (need experimental justification):
  1. The skymap of sources is **stationary** over the exp. lifetime.

     Neglect transients. Their effect will depend on the timescales involved.

  2. The observed events are **independent**.

     The probability of observing an event at a given position depends on the source skymap, but not on previous events already observed.

The statistics of the observable $\tilde{C}_{\ell,N}$ are exactly solvable in this case.



Statistical Mean: Events Relate to Sources!

- The average measurement of $\tilde{C}_{\ell,N}$ from a random sample:

$$\langle \tilde{C}_{\ell,N} \rangle = \frac{4\pi}{N} + \left(1 - \frac{1}{N}\right) \tilde{C}_\ell$$

$\tilde{C}_\ell$ is now APS of source skymap, convolved with instrument PSF.

  - Angular power spectrum of events is a *biased* estimator of the source distribution.

- Therefore, an unbiased estimator $\hat{\tilde{C}}_{\ell,N}$ with $\left\langle \hat{\tilde{C}}_{\ell,N} \right\rangle = \tilde{C}_\ell$:

$$\hat{\tilde{C}}_{\ell,N} = \frac{1}{1 - \frac{1}{N}} \left[\tilde{C}_{\ell,N} - \frac{4\pi}{N} \right] = \frac{4\pi}{N(N-1)} \sum_i \sum_{j \neq i} P_\ell(\boldsymbol{n}_i \cdot \boldsymbol{n}_j)$$

  - In agreement with other existing methods!

Exact Statistical Covariance of $\hat{\hat{C}}_{\ell,N}$

$$\mathrm{Cov}\left[\hat{\hat{C}}_{\ell_1,N},\hat{\hat{C}}_{\ell_2,N}\right]=\frac{(4\pi)^2}{N(N-1)}\left\{2\left[\frac{\delta_{\ell_1,\ell_2}}{2\ell_1+1}+\tilde{C}^{(2)}_{\ell_1\ell_2}-\frac{\tilde{C}_{\ell_1}\tilde{C}_{\ell_2}}{(4\pi)^2}\right]\right.$$
$$\left.+4(N-2)\left[\frac{\delta_{\ell_1,\ell_2}}{2\ell_1+1}\frac{\tilde{C}_{\ell_1}}{4\pi}+\frac{\tilde{C}^{(3)}_{\ell_1\ell_2}}{4\pi}-\frac{\tilde{C}_{\ell_1}\tilde{C}_{\ell_2}}{(4\pi)^2}\right]\right\}$$

$$\tilde{C}^{(2)}_{\ell_1\ell_2}=\sum_{\ell'=|\ell_1-\ell_2|}^{\ell_1+\ell_2}\frac{2\ell'+1}{4\pi}\begin{pmatrix}\ell_1 & \ell_2 & \ell'\\ 0 & 0 & 0\end{pmatrix}^2\tilde{C}_{\ell'}$$

$$\tilde{C}^{(3)}_{\ell_1\ell_2}=\frac{1}{\sqrt{(2\ell_1+1)(2\ell_2+1)}}\sum_{\ell'=|\ell_1-\ell_2|}^{\ell_1+\ell_2}\sqrt{\frac{2\ell'+1}{4\pi}}\begin{pmatrix}\ell_1 & \ell_2 & \ell'\\ 0 & 0 & 0\end{pmatrix}\tilde{B}_{\ell_1\ell_2\ell'}$$

Analytic Work Generated
Higher Order Angular Spectra

$$\tilde{C}^{(2)}_{\ell_1\ell_2} = \sum_{\ell'=|\ell_1-\ell_2|}^{\ell_1+\ell_2} \frac{2\ell'+1}{4\pi}\begin{pmatrix}\ell_1 & \ell_2 & \ell' \\ 0 & 0 & 0\end{pmatrix}^2 \tilde{C}_{\ell'}$$

$$\tilde{C}^{(3)}_{\ell_1\ell_2} = \frac{1}{\sqrt{(2\ell_1+1)(2\ell_2+1)}} \sum_{\ell'=|\ell_1-\ell_2|}^{\ell_1+\ell_2} \sqrt{\frac{2\ell'+1}{4\pi}}\begin{pmatrix}\ell_1 & \ell_2 & \ell' \\ 0 & 0 & 0\end{pmatrix} \tilde{B}_{\ell_1\ell_2\ell'}$$

$$\tilde{C}^{(4)}_{\ell_1\ell_2} = \tilde{C}_{\ell_1}\tilde{C}_{\ell_2}$$

I know two ways to see that $\tilde{C}_\ell$ is the first order angular spectrum, and that these comprise the **complete** set of 2nd order spectra.

Higher Order Spectra: Tensor Picture

- First and Second Rank Spherical Harmonic Transforms of S:

$$\tilde{a}_{\ell m} = \int d\boldsymbol{n}\; Y^*_{\ell m}(\boldsymbol{n})\, S(\boldsymbol{n}), \qquad \tilde{a}_{\ell_1 m_1 \ell_2 m_2} = \int d\boldsymbol{n}\, Y^*_{\ell_1 m_1}(\boldsymbol{n}) Y^*_{\ell_2 m_2}(\boldsymbol{n})\, S(\boldsymbol{n})$$

- Raised Azimuthal Indices generated by $Y_\ell{}^m = (-1)^m\, Y^*_{\ell,-m}$:

$$\tilde{a}_{\ell_1 m_1 \ell_1}{}^{m_2} = \int d\boldsymbol{n}\, Y^*_{\ell m_1}(\boldsymbol{n}) Y_\ell{}^{m_2}(\boldsymbol{n})\, S(\boldsymbol{n}) = (-1)^{m_2}\, \tilde{a}_{\ell_1, m_1, \ell_2, -m_2}$$

- Create rank 0 (rotation invariant) tensors by contracting azimuthal indices:

$$\tilde{C}_\ell = \frac{1}{2\ell + 1} \sum_{m=-\ell}^{\ell} \tilde{a}_\ell{}^m \tilde{a}_{\ell m}$$

Higher Order Spectra: Tensor Picture

- All possible rank 0 tensors from rank 1 and 2 transforms.

$$\tilde{C}^{(2)}_{\ell_1\ell_2} = \frac{1}{(2\ell_1+1)(2\ell_2+1)} \sum_{m_1,m_2} \tilde{a}_{\ell_1}{}^{m_1}{}_{\ell_2}{}^{m_2} \, \tilde{a}_{\ell_1 m_1 \ell_2 m_2}$$

$$\tilde{C}^{(3)}_{\ell_1\ell_2} = \frac{1}{(2\ell_1+1)(2\ell_2+1)} \sum_{m_1,m_2} \tilde{a}_{\ell_1}{}^{m_1}{}_{\ell_2}{}^{m_2} \, \tilde{a}_{\ell_1 m_1} \tilde{a}_{\ell_2 m_2}$$

$$\begin{aligned} \tilde{C}^{(4)}_{\ell_1\ell_2} &= \tilde{C}_{\ell_1}\tilde{C}_{\ell_2} \\ &= \frac{1}{(2\ell_1+1)(2\ell_2+1)} \sum_{m_1,m_2} \tilde{a}_{\ell_1}{}^{m_1} \tilde{a}_{\ell_1 m_1} \tilde{a}_{\ell_2}{}^{m_2} \tilde{a}_{\ell_2 m_2} \end{aligned}$$

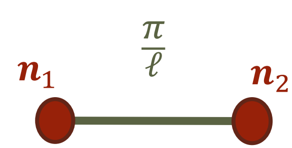
Higher Order Spectra: Field Theory Pic.

- Use the Spherical Harmonic Addition Theorem:

$$\frac{1}{2\ell+1}\sum_m Y_\ell^{\ m}(\boldsymbol{n}_1)\, Y^*_{\ell m}(\boldsymbol{n}_2) = \frac{1}{4\pi} P_\ell(\boldsymbol{n}_1 \cdot \boldsymbol{n}_2)$$

- Angular Power Spectrum is like 2 field configurations connected by a "correlator".

$$\tilde{C}_\ell = 4\pi \int \frac{d\boldsymbol{n}_1}{4\pi}\frac{d\boldsymbol{n}_2}{4\pi}\, S(\boldsymbol{n}_1) P_\ell(\boldsymbol{n}_1 \cdot \boldsymbol{n}_2)\, S(\boldsymbol{n}_2)$$



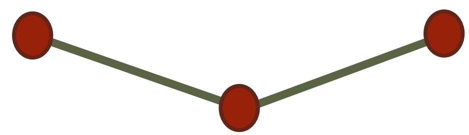
Higher Order Spectra: Field Theory Pic.

- All possible diagrams with 2 correlators.

$$\tilde{C}^{(2)}_{\ell_1\ell_2} = \int \frac{d\boldsymbol{n}_1}{4\pi}\frac{d\boldsymbol{n}_2}{4\pi}\, S(\boldsymbol{n}_1)P_{\ell_1}(\boldsymbol{n}_1\cdot\boldsymbol{n}_2)P_{\ell_2}(\boldsymbol{n}_1\cdot\boldsymbol{n}_2)S(\boldsymbol{n}_2)$$

"Composite Angular Power Spectrum"

$$\tilde{C}^{(3)}_{\ell_1\ell_2} = 4\pi \int \frac{d\boldsymbol{n}_1}{4\pi}\frac{d\boldsymbol{n}_2}{4\pi}\frac{d\boldsymbol{n}_3}{4\pi}\, S(\boldsymbol{n}_1)P_{\ell_1}(\boldsymbol{n}_1\cdot\boldsymbol{n}_2)\, S(\boldsymbol{n}_2)P_{\ell_2}(\boldsymbol{n}_2\cdot\boldsymbol{n}_3)\, S(\boldsymbol{n}_3)$$



"Open Angular Bispectrum"

$$\tilde{C}^{(4)}_{\ell_1\ell_2} = \tilde{C}_{\ell_1}\tilde{C}_{\ell_2}$$

"Disjoint Angular Trispectrum"

Unbiased Estimators from N Events

$$\hat{\hat{C}}^{(2)}_{\ell_1\ell_2,N} = \frac{1}{N(N-1)}\sum_{i_1}\sum_{i_2\neq i_1} P_{\ell_1}(\boldsymbol{n}_{i_1}\cdot\boldsymbol{n}_{i_2})P_{\ell_2}(\boldsymbol{n}_{i_1}\cdot\boldsymbol{n}_{i_2}) - \frac{\delta_{\ell_1\ell_2}}{2\ell_1+1}$$

$$\hat{\hat{C}}^{(3)}_{\ell_1\ell_2,N} = \frac{4\pi}{N(N-1)(N-2)}\sum_{i_1}\sum_{i_2\neq i_1}\sum_{\substack{i_3\neq i_2\\ i_3\neq i_1}} P_{\ell_1}(\boldsymbol{n}_{i_1}\cdot\boldsymbol{n}_{i_2})P_{\ell_2}(\boldsymbol{n}_{i_2}\cdot\boldsymbol{n}_{i_3}) - \frac{\delta_{\ell_1\ell_2}}{2\ell_1+1}\hat{\hat{C}}_{\ell_1,N}$$

$$\hat{\hat{C}}^{(4)}_{\ell_1\ell_2,N} = \frac{(4\pi)^2}{N(N-1)(N-2)(N-3)}\sum_{i_1}\sum_{i_2\neq i_1}\sum_{\substack{i_3\neq i_2\\ i_3\neq i_1}}\sum_{\substack{i_4\neq i_3\\ i_4\neq i_2\\ i_4\neq i_1}} P_{\ell_1}(\boldsymbol{n}_{i_1}\cdot\boldsymbol{n}_{i_2})P_{\ell_2}(\boldsymbol{n}_{i_3}\cdot\boldsymbol{n}_{i_4})$$

Statistical Covariance of $\hat{\hat{C}}_{\ell,N}$ $(N \gg 1)$

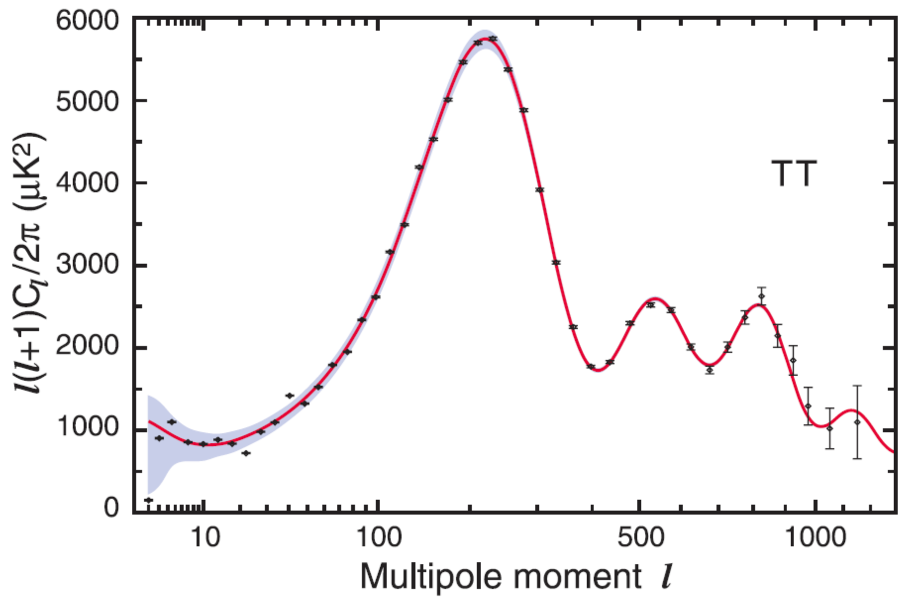
$$\mathrm{Cov}\left[\hat{\hat{C}}_{\ell_1,N}, \hat{\hat{C}}_{\ell_2,N}\right] = (4\pi)^2 \left\{ \frac{2}{N^2}\left[\frac{\delta_{\ell_1,\ell_2}}{2\ell_1+1} + \tilde{C}^{(2)}_{\ell_1\ell_2} - \frac{\tilde{C}_{\ell_1}\tilde{C}_{\ell_2}}{(4\pi)^2}\right] + \frac{4}{N}\left[\frac{\delta_{\ell_1,\ell_2}}{2\ell_1+1}\frac{\tilde{C}_{\ell_1}}{4\pi} + \frac{\tilde{C}^{(3)}_{\ell_1\ell_2}}{4\pi} - \frac{\tilde{C}_{\ell_1}\tilde{C}_{\ell_2}}{(4\pi)^2}\right]\right\}$$

- If higher-order spectra are neglected:
  - the covariance is diagonal—each multipole measurement is independent.
  - call this C_ℓ-only statistical uncertainty.

$$\mathcal{V}\mathrm{ar}\left[\hat{\hat{C}}_{\ell,N}\right] = \frac{2}{2\ell+1}\left[\left(\frac{4\pi}{N}\right)^2 + 2\left(\frac{4\pi}{N}\right)\tilde{C}_\ell\right] \qquad (C_\ell\text{-only})$$

Good Agreement with WMAP Data

$$\mathcal{V}\mathrm{ar}\left[\hat{\tilde{C}}_{\ell,N}\right] = \frac{2}{2\ell+1}\left[\left(\frac{4\pi}{N}\right)^2 + 2\left(\frac{4\pi}{N}\right)\tilde{C}_\ell\right] \qquad (C_\ell\text{-only})$$

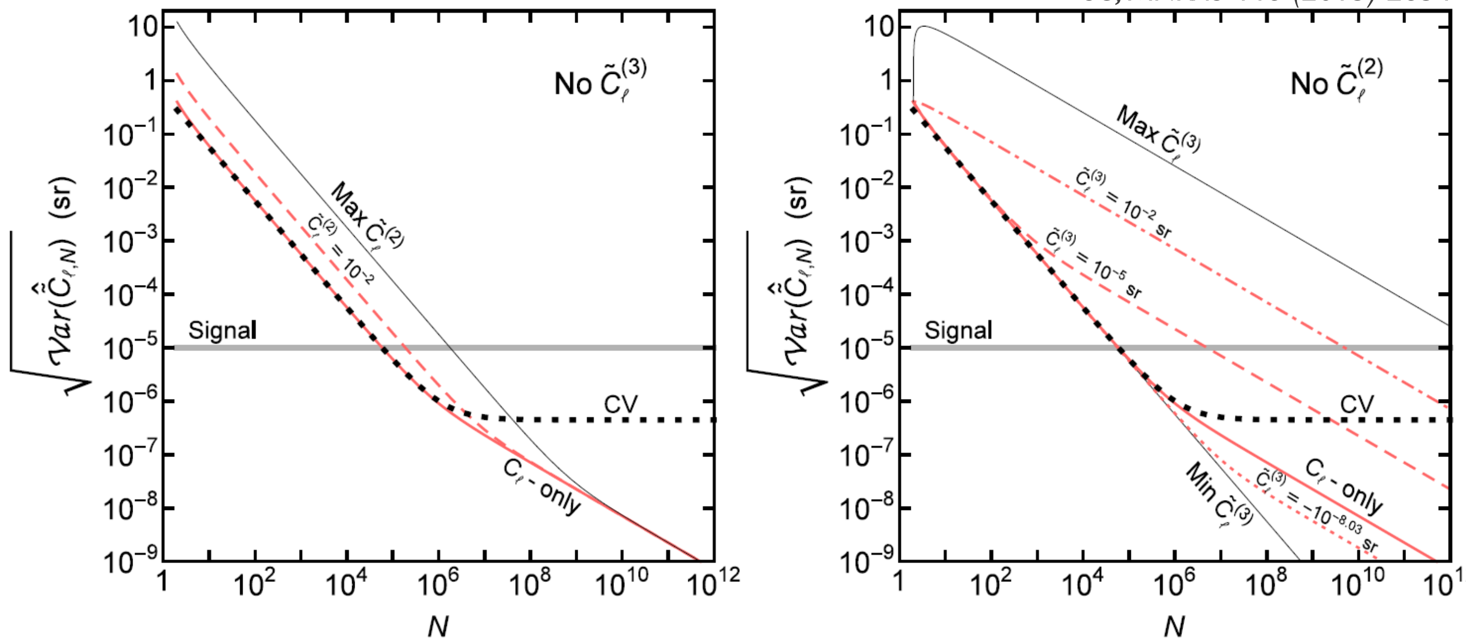


WMAP Collaboration, Astrophys.J.Suppl. 208 (2013) 20 & PTEP 2014 (2014) 6, 06B102

Fig. 5 Nine-year angular power spectrum of the CMB temperature (adapted from [37]). While we measure C_ℓ at each ℓ in $2 \le \ell \le 1200$, the points with error bars show the binned values of C_ℓ for clarity. The error bars show the standard deviation of C_ℓ from instrumental noise, $[2(2C_\ell N_\ell + N_\ell^2)/(2\ell+1)f_{\mathrm{sky},\ell}^2]^{1/2}$. The shaded area shows the standard deviation from the cosmic variance term, $[2C_\ell^2/(2\ell+1)f_{\mathrm{sky},\ell}^2]^{1/2}$ (except at very low ℓ where the 68% CL from the full non-Gaussian posterior probability is shown). The solid line shows the theoretical curve of the best-fit ΛCDM cosmological model.

The New Error Terms Can Be Important

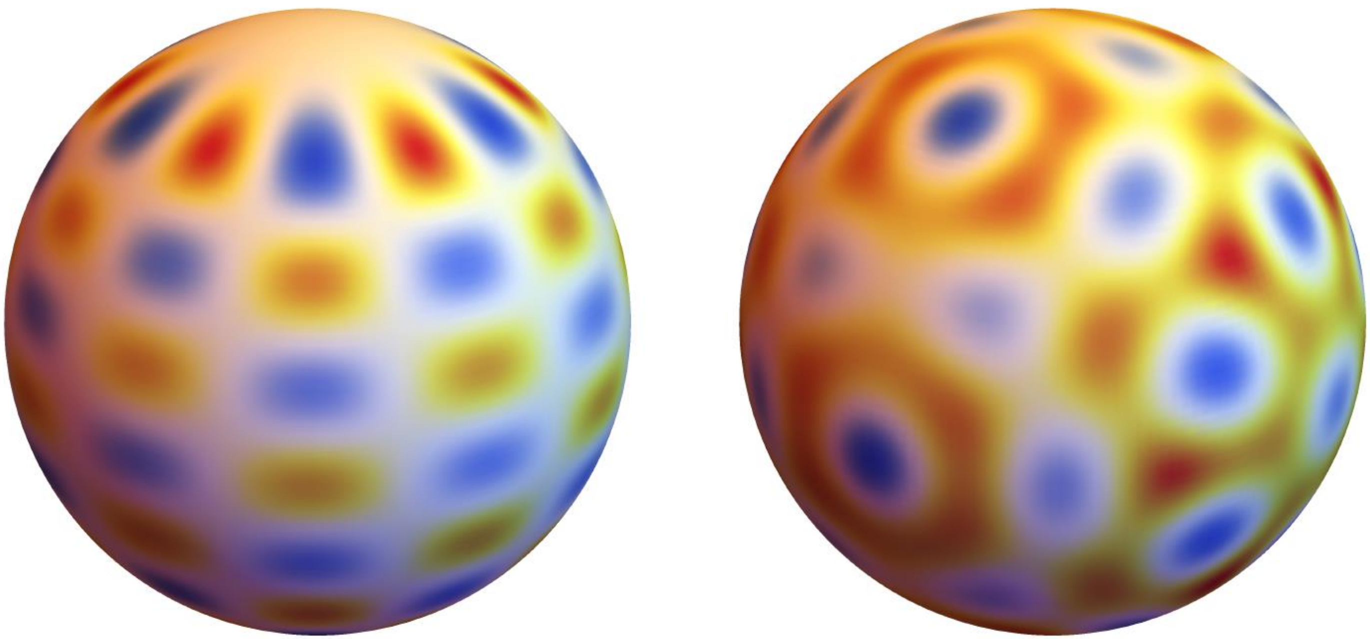
SC, MNRAS 448 (2015) 2854



- Example uncertainty evolution at $\ell = 500$ with $\tilde{C}_\ell = 10^{-5}$ sr.
- But is this a real effect? Does a distribution with bispectrum have a different power spectrum uncertainty than one without bispectrum?

Test with Monte-Carlo Sampling

SC, MNRAS 448 (2015) 2854

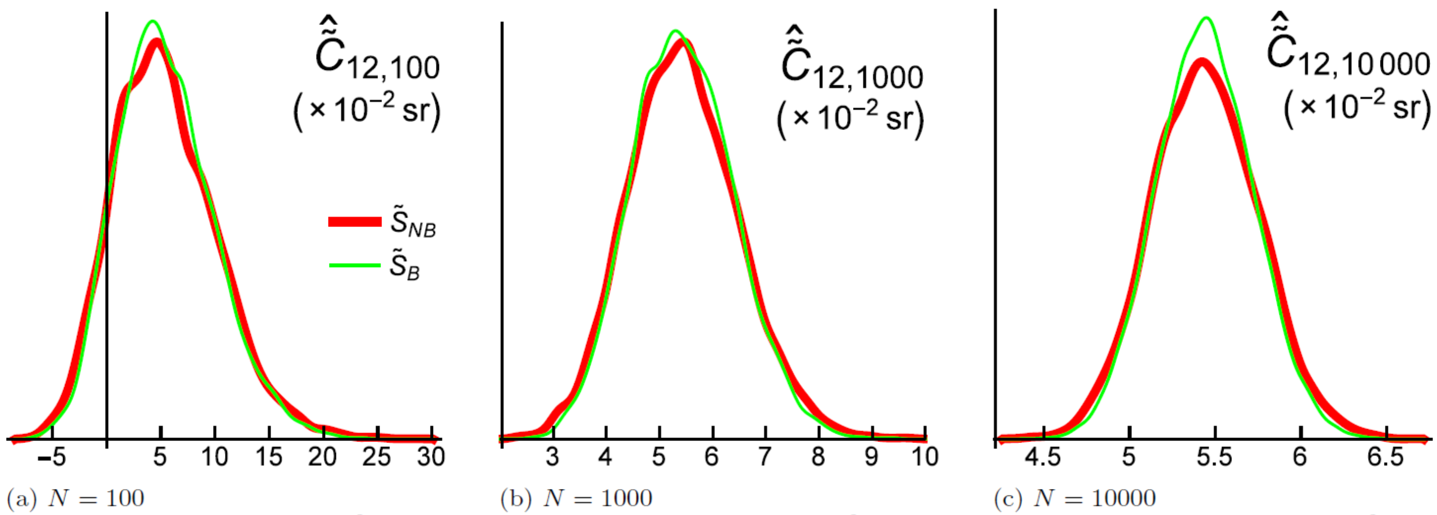


(a) $\tilde{S}_{NB}(\mathbf{n})$ (b) $\tilde{S}_{B}(\mathbf{n})$

$\tilde{C}_\ell = (0.0544\ \mathrm{sr})\ \delta_{\ell,12}$	$\tilde{C}_\ell = (0.0544\ \mathrm{sr})\ \delta_{\ell,12}$
$\tilde{C}^{(3)}_{\ell\ell} = 0$	$\tilde{C}^{(3)}_{\ell\ell} = (-0.000413\ \mathrm{sr})\ \delta_{\ell,12}$

$\hat{\tilde{C}}_{\ell,N}$ Distribution of 10 000 Samplings

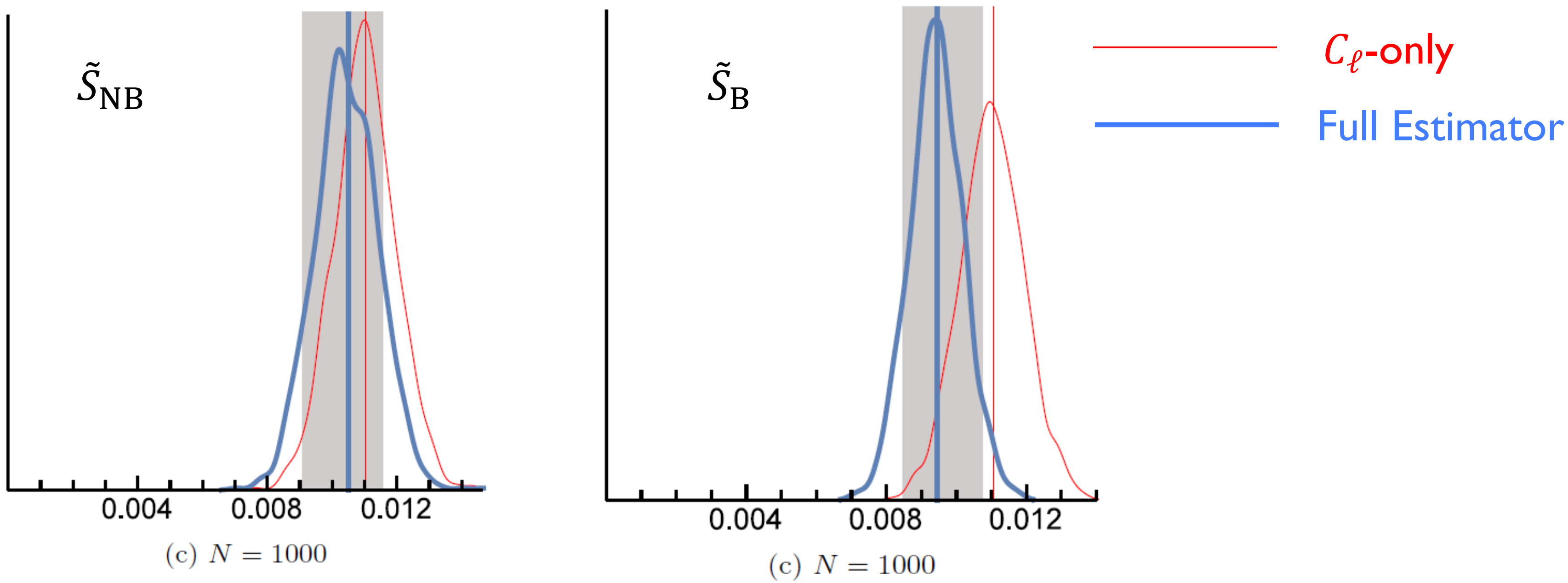
SC, MNRAS 448 (2015) 2854



- Low counts gives very wide distribution. Shot noise subtraction can give negative power spectrum estimates.
- At high counts, the distribution becomes narrow, and the distribution with negative bispectrum is visibly narrower.

$\sigma_{\hat{\hat{C}}_{\ell,N}}$ Distribution of 10 000 Samplings

SC, MNRAS 448 (2015) 2854



- The negative bispectrum does indeed appear to lower the variance of the power spectrum measurement.
- Even the distribution without bispectrum is affected by the other higher-order spectra, but those effects are small and unresolved in this example.

Conclusions

- A new analytic error analysis of angular power spectra of points is presented. This is a natural analysis to carry out with IceCube data.
- The unbiased estimator of the source's angular power spectrum is in agreement with usual estimates.
- The uncertainty has the usual shot noise and first order signal contributions, but gives new higher order anisotropy contributions.
- These results do not assume Gaussianity of signal/sources.
  - Results apply to any event distribution from stationary sources.
- These results allow for realistic estimates of the data requirements for distinguishing source models through angular distributions.

Extra Slides

A Popular Measure of Angular Distribution: The Angular Power Spectrum

Intensity Angular Power Spectrum C_ℓ

$$I(E,\boldsymbol{n}) - \langle I(E)\rangle = \sum_{\ell,m} a_{\ell m}(E) Y_\ell^m(\boldsymbol{n}) \qquad C_\ell(E) = \frac{1}{2\ell+1}\sum_m |a_{\ell m}(E)|^2$$

- Absolute intensity fluctuations.
- Monotonically increases as sources are added.

***Fluctuation* Angular Power Spectrum $\widetilde{C_\ell}$**

$$\frac{I(E,\boldsymbol{n}) - \langle I(E)\rangle}{\langle I(E)\rangle} = \sum_{\ell,m} \tilde{a}_{\ell m}(E) Y_\ell^m(\boldsymbol{n}) \qquad \widetilde{C_\ell}(E) = \frac{1}{2\ell+1}\sum_m |\tilde{a}_{\ell m}(E)|^2$$

- Relative intensity fluctuations.
- Constant for universal spectrum sources at fixed redshift.

Special Case: Pure Isotropic Source

- Receive N events at uniformly random positions.

$$\tilde{a}_{\ell m,N} = \frac{4\pi}{N} \sum_{i=1}^{N} Y_{\ell m}^{*}(\hat{n}_i) \qquad \tilde{C}_{\ell,N} = \frac{1}{2\ell+1} \sum_{m=-\ell}^{\ell} \left|\tilde{a}_{\ell m,N}\right|^2$$

$$\langle \tilde{C}_{\ell,N} \rangle = \tilde{C}_{P,N} = \frac{4\pi}{N}$$

Shot noise/Poisson noise.

$$\sigma_{\tilde{C}_{\ell,N}} = \sqrt{\frac{2}{2\ell+1} \frac{4\pi}{N}}$$

Error Estimate with Anisotropic Source

- Lesson from CMB: Cosmic Variance

- The dominant statistical uncertainty in CMB anisotropy.
  - Cosmic Variance ⟺ Unknown Initial Conditions

- Assuming the signal is randomly Gaussian distributed, then our estimator for $\tilde{C}_\ell$ is the maximum likelihood estimator with uncertainty:

$$\sigma_{\tilde{C}_\ell} = \sqrt{\frac{2}{2\ell + 1}}\tilde{C}_\ell$$

"Rule of Thumb" Stat. Uncertainty Est.

- Angular power spectrum from "events".
- Assume sources are approximately Gaussian distributed.
- Shot noise is a bias to be subtracted from estimator.

$$\hat{\hat{C}}_{\ell,N} = \frac{1}{2\ell+1} \sum_{m=-\ell}^{\ell} \left| \frac{4\pi}{N} \sum_{i=1}^{N} Y^*_{\ell m}(\boldsymbol{n}_i) \right|^2 - \frac{4\pi}{N}$$

$$\sigma_{\hat{\hat{C}}_{\ell,N}} = \sqrt{\frac{2}{2\ell+1}} \left(\frac{4\pi}{N} + \tilde{C}_\ell \right)$$

Knox, PRD52, 4307 (1995)

- The goal is to check these standard estimates.

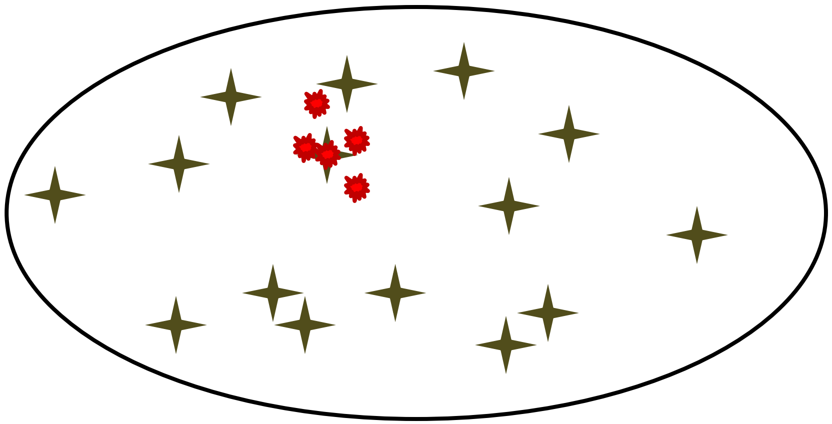
Improving Our Understanding of the Statistical Variance

- Some conceptual difficulties with using the cosmic variance as we did.
  - Cosmic variance is a theoretical error, which applies when making physical inferences about our models based on data.
  - The angular power spectrum measurement should be able to be made independently of any model.
  - We should not need to assume the signal is Gaussian-distributed.

- Investigations have led to a new formula for the model-independent statistical variance of the angular power spectrum of events from a background distribution.

Strategy for Calculation

Consider each event observed at position $\hat{n}'$ but originated from position $\hat{n}$.



1) For fixed source positions $\hat{n}_i$, average over event position $\hat{n}_i{}'$, via the instrument point spread function.

   Result of this step: what is being measured is the sky map convolved with the instrument PSF.

2) Average the N events source positions, weighted by the skymap.

Compare to Gaussian Cosmic Variance

- Old method with shot noise + Gaussian cosmic variance:

$$\sigma^2_{\hat{\tilde{C}}_{\ell,N}} = \frac{2}{2\ell+1}\left(\frac{4\pi}{N} + \tilde{C}_\ell\right)^2$$
$$\simeq \left(\frac{4\pi}{N}\right)^2 \left[\frac{2}{2\ell+1} + \frac{4N}{2\ell+1}\frac{\tilde{C}_\ell}{4\pi} + \frac{2N^2}{2\ell+1}\left(\frac{\tilde{C}_\ell}{4\pi}\right)^2\right]$$

- New variance formula:

$$\sigma^2_{\hat{\tilde{C}}_{\ell,N}} \simeq \left(\frac{4\pi}{N}\right)^2 \left[\frac{2}{2\ell+1} + 2\tilde{C}_\ell^{(2)} + \frac{4N}{2\ell+1}\frac{\tilde{C}_\ell}{4\pi} + 4N\frac{\tilde{C}_\ell^{(3)}}{4\pi} - 4N\left(\frac{\tilde{C}_\ell}{4\pi}\right)^2\right]$$

- The new formula agrees surprisingly well with the traditional estimate, with dominant contributions for a weak signal in precise agreement.
- New terms important at large N. Note no N-independent terms!

Gaussian-Distributed Sky Map

- Our results do not assume Gaussianity.
- If the sky map is Gaussian, then higher order spectra are determined from $\tilde{C}_\ell$ as follows:

$$\left\langle \tilde{C}_\ell^{(2)} \right\rangle = \sum_{\ell'=0}^{2\ell} \frac{2\ell' + 1}{4\pi} \begin{pmatrix} \ell & \ell & \ell' \\ 0 & 0 & 0 \end{pmatrix}^2 \langle \tilde{C}_{\ell'} \rangle$$

$$\left\langle \tilde{C}_\ell^{(3)} \right\rangle = 0$$

$$\left\langle \tilde{C}_\ell^{(4)} \right\rangle = \frac{2\ell + 3}{2\ell + 1} \langle \tilde{C}_\ell \rangle^2$$

Consequences of Findings

- Experiments using Monte Carlo to estimate error already take into account these new effects automatically.
- Experiments using Gaussian Cosmic Variance may be missing higher orders in the uncertainty of angular power.
  - Fermi-LAT anisotropy measurement should check estimators of these terms for possible corrections to their uncertainties.
  - Small χ^2 suggests either their errors should be smaller (possibly due to some more subtle effects) or energy bins are somehow correlated.
- This error analysis must also take into account effects of:
  - non-uniform exposure,
  - sky masking,
  - other observational bias or instrumental effects.